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2003 J. Phys. A: Math. Gen. 36 L263

(http://iopscience.iop.org/0305-4470/36/17/102)

INSTITUTE OF PHYSICS PUBLISHING — JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) L263–L272 — PII: S0305-4470(03)61028-0

LETTER TO THE EDITOR

${}_{10}E_9$ solution to the elliptic Painlevé equation

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Received 18 March 2003
Published 16 April 2003
Online at stacks.iop.org/JPhysA/36/L263

Abstract

A τ function formalism for Sakai's elliptic Painlevé equation is presented. This establishes the equivalence between the two formulations by Sakai and by Ohta–Ramani–Grammaticos. We also give a simple geometric description of the elliptic Painlevé equation as a non-autonomous deformation of the addition formula on elliptic curves. By using these formulations, we construct a particular solution of the elliptic Painlevé equation expressed in terms of the elliptic hypergeometric function ${}_{10}E_9$.

PACS numbers: 02.30.Ks, 02.30.Gp, 02.40.Re
Mathematics Subject Classification: 39A20, 33D67, 33E17, 14E07, 14H52

Dedicated to Professor Kyoichi Takano on his sixtieth birthday.

1. Introduction

The elliptic Painlevé equation proposed by Sakai [1] is located on the top of Painlevé and discrete Painlevé equations. It is a second-order nonlinear difference equation with affine Weyl group symmetry of type $E_8^{(1)}$, and regarded as the discrete dynamical system on a rational surface obtained by blowing-up at nine points on $\mathbb{P}^2$. All the Painlevé and discrete Painlevé equations are derived by degeneration from the elliptic Painlevé equation.

On the other hand, it is well known that Painlevé and discrete Painlevé equations admit special cases which are reducible to the Riccati equations. From them we obtain particular solutions described by special functions of hypergeometric type [2–4]. We call such solutions the *Riccati solutions* or the *hypergeometric solutions*. A natural question is what kind of function of hypergeometric type arises as such a particular solution to the elliptic Painlevé equation. The purpose of this letter is to present an answer to this question: it is the elliptic hypergeometric function ${}_{10}E_9$ introduced by Frenkel–Turaev [5], and studied further by Spiridonov–Zhedanov [6].

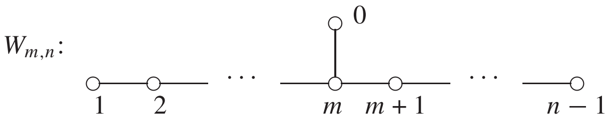
In section 2, we introduce a framework of τ functions for the elliptic Painlevé equation, and describe explicitly the birational action of the affine Weyl group $W\big(E_8^{(1)}\big)$ on the level of the τ functions. We also present bilinear equations to be satisfied by the τ functions. In section 3, we give an explicit form of the elliptic Painlevé equation, and construct a particular solution described by ${}_{10}E_9$. We also present a simple geometric description for the elliptic Painlevé equation in section 4, and give another method for constructing the hypergeometric solution.

2. τ function

Let us recall briefly Sakai's formulation of the elliptic Painlevé equation [1]. This approach, based on the geometry of rational surfaces, was initiated by Okamoto [7] and pursued further by Takano (see for instance [8]). Consider the configuration space $\mathcal{M}_{m,n}$ $(0 < m < n)$ of n points $P_i = (x_{1i} : x_{2i} : \ldots : x_{mi})$ $(1 \leqslant i \leqslant n)$ on $\mathbb{P}^{m-1}$:

$$\mathcal{M}_{m,n} = \mathrm{GL}(m)\backslash \left\{ \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix} \right\} / (\mathbb{C}^\times)^n. \tag{1}$$

Let $W_{m,n}$ be the Weyl group generated by simple reflections s_i $(i = 0, 1, \ldots, n-1)$ corresponding to the following Dynkin–Coxeter diagram:



There is a birational action of the Weyl group $W_{m,n}$ on the configuration space $\mathcal{M}_{m,n}$ [9]. In the case of $(m, n) = (3, 10)$, the Weyl group $W_{3,10}$ contains a translation subgroup $\mathbb{Z}^8 \subset W\big(E_8^{(1)}\big) = W_{3,9} \subset W_{3,10}$. The birational action of the translation is nothing but the elliptic Painlevé equation [1]; the points $P_1, \ldots, P_9$ play the role of parameters, while the last point P_{10} is the dependent variable [10].

Let $P = \mathbb{Z}e_0 \oplus \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n$ be the lattice with the metric defined by

$$\left\langle \sum_{i=0}^{n} x_i e_i, \sum_{i=0}^{n} y_i e_i \right\rangle = -(m-2)x_0 y_0 + \sum_{i=1}^{n} x_i y_i. \tag{2}$$

We consider the root lattice $Q = \mathbb{Z}\alpha_0 \oplus \cdots \oplus \mathbb{Z}\alpha_{n-1} \subset P$ generated by

$$\alpha_i = e_i - e_{i+1} \quad (1 \leqslant i \leqslant n-1) \qquad \alpha_0 = e_0 - e_1 - e_2 - \cdots - e_m. \tag{3}$$

The standard formula of the reflection with respect to α_i $(i = 0, \ldots, n-1)$,

$$s_i(L) = L - \langle \alpha_i, L \rangle \alpha_i \qquad (i = 0, \ldots, n-1) \tag{4}$$

gives the action of the Weyl group $W_{m,n}$ on the lattice P. In the Painlevé context, vectors $v \in \sum_{i=0}^{n} \mathbb{C}e_i$ play the role of parameters (or independent variable), whose coordinates are denoted by $\varepsilon_i = \langle e_i, v \rangle$ $(i = 0, \ldots, n)$. Then $W_{m,n}$ acts on them as follows:

$$\begin{aligned} s_0(\varepsilon_0) &= (m-1)\varepsilon_0 - \varepsilon_1 - \cdots - \varepsilon_m \\ s_0(\varepsilon_i) &= (m-2)\varepsilon_0 - \varepsilon_1 - \cdots - \widehat{\varepsilon_i} - \cdots - \varepsilon_m \qquad (i = 1, \ldots, m) \end{aligned}$$

$$
\begin{aligned}
&s_0(\varepsilon_j) = \varepsilon_j \qquad (j = m+1, \ldots, n) \\
&s_i(\varepsilon_i) = \varepsilon_{i+1} \qquad s_i(\varepsilon_{i+1}) = \varepsilon_i \qquad (i = 1, \ldots, n-1) \\
&s_i(\varepsilon_j) = \varepsilon_j \qquad \text{(otherwise).}
\end{aligned} \tag{5}
$$

Let K be the field of entire meromorphic functions in $(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n)$, and consider the field $K(f)$ of rational functions in the indeterminates $f = (f_1, \ldots, f_m)$ with coefficients in K. Let $\mathcal{R} = K(f)(\tau_1, \ldots, \tau_n)$ be the field of rational functions in $(\tau_1, \ldots, \tau_n)$ over $K(f)$. Then one can construct a birational representation of $W_{m,n}$ on $\mathcal{R}$ as follows:

Theorem 2.1. *We define the birational actions of s_i $(i = 0, \ldots, n-1)$ on $\mathcal{R}$ as follows:*

$$
\begin{aligned}
&s_0(\tau_i) = f_i\tau_i \quad (i = 1, \ldots, m) \qquad s_0(\tau_j) = \tau_j \quad (j = m+1, \ldots, n) \\
&s_i(\tau_i) = \tau_{i+1} \qquad s_i(\tau_{i+1}) = \tau_i \qquad (i = 1, \ldots, n-1) \\
&s_i(\tau_j) = \tau_j \qquad (\textit{otherwise})
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
&s_0(f_i) = \frac{1}{f_i} \qquad (i = 1, \ldots, m) \\
&s_i(f_i) = f_{i+1} \qquad s_i(f_{i+1}) = f_i \qquad (i = 1, \ldots, m-1) \\
&s_m(f_i) = \frac{\tau_m}{\tau_{m+1}}(c_{i,m}f_i + c_{m,i}f_m) \qquad (i = 1, \ldots, m-1) \\
&s_m(f_m) = \frac{\tau_m}{\tau_{m+1}} f_m \\
&s_i(f_j) = f_j \qquad (\textit{otherwise})
\end{aligned} \tag{7}
$$

where $c_{i,j} = \frac{[\varepsilon_i - \varepsilon_{m+1}][\alpha_0 + \varepsilon_j - \varepsilon_{m+1}]}{[\varepsilon_i - \varepsilon_j][\alpha_0]}$ *and* $[x] = \vartheta_{11}(x)$ *is the odd theta function. Then* $\langle s_0, \ldots, s_{n-1}\rangle$ *forms the Weyl group* $W_{m,n}$.

Proof. By direct computation, the Coxeter relation $(s_is_j)^2 = 1$ or $(s_is_j)^3 = 1$ follows from the Riemann relation

$$
[a+b][a-b][c+z][c-z] + (abc \text{ cyclic}) = 0 \tag{8}
$$

and $[-x] = -[x]$. □

Let M be the Weyl group orbit $M = W_{m,n} . e_1 \subset P$. For any $\Lambda \in M$, choose $w \in W_{m,n}$ such that $w \cdot e_1 = \Lambda$ and define the τ function $\tau(\Lambda)$ as

$$
\tau(\Lambda) = w(\tau_1). \tag{9}
$$

This definition is independent of the choice of w and we have

$$
\begin{aligned}
&w(\tau(\Lambda)) = \tau(w . \Lambda) \qquad (w \in W_{m,n}, \Lambda \in M) \\
&\tau(e_i) = \tau_i \qquad (i = 1, \ldots, n).
\end{aligned} \tag{10}
$$

Then, one can derive the following bilinear difference equations of Hirota–Miwa type for the τ functions:

Proposition 2.2. *For any mutually distinct indices $1 \leqslant i, j, k, l_1, \ldots, l_{m-2} \leqslant n$, we have*

$$
\begin{aligned}
&[\varepsilon_j - \varepsilon_k][L - \varepsilon_j - \varepsilon_k]\tau(e_i)\tau(L - e_i) + [\varepsilon_k - \varepsilon_i][L - \varepsilon_k - \varepsilon_i]\tau(e_j)\tau(L - e_j) \\
&\qquad + [\varepsilon_i - \varepsilon_j][L - \varepsilon_i - \varepsilon_j]\tau(e_k)\tau(L - e_k) = 0
\end{aligned} \tag{11}
$$

where $L = e_0 - e_{l_1} - \cdots - e_{l_{m-2}}$.

Proof. Using the relation $f_i = s_0(\tau_i)/\tau_i$, the formula for $s_m(f_i)$ can be written in terms of τ as

$$\tau_{m+1} s_m s_0(\tau_i) = c_{i,m} \tau_m s_0(\tau_i) + c_{m,i} \tau_i s_0(\tau_m). \tag{12}$$

This equation is a special case of equation (11) where $(i, j, k) = (i, m, m + 1)$ and $L = \alpha_0 + e_i + e_m$. Other cases can be obtained by the $\mathfrak{S}_n$-symmetry. □

Moreover, by reversing the above argument, we have the following:

Theorem 2.3. *For any family of functions $\tau(\Lambda)$ ($\Lambda \in M$), we define variables f_i by $f_i = \tau(s_0 . e_i)/\tau(e_i)$ for $i = 1, \ldots, m$. Then the actions of the Weyl group $W_{m,n}$ on M are consistent with the transformations (6) and (7) of f_i and $\tau_j = \tau(e_j)$ ($i = 1, \ldots, m$, $j = 1, \ldots, n$) if and only if the bilinear relations (11) are satisfied.*

In what follows, we consider the case $(m, n) = (3, 9)$ of nine points $P_1, \ldots, P_9$ in $\mathbb{P}^2$, together with an additional point P_{10} playing the role of the dependent variable of the elliptic Painlevé equation. The formulation of the τ functions described above is relevant to the following parametrization of the elliptic curve C_0 passing through the nine points:

$$P(u) = \left(\frac{[\varepsilon_0 - \varepsilon_2 - \varepsilon_3 - u]}{[\varepsilon_1 - u]} : \frac{[\varepsilon_0 - \varepsilon_3 - \varepsilon_1 - u]}{[\varepsilon_2 - u]} : \frac{[\varepsilon_0 - \varepsilon_1 - \varepsilon_2 - u]}{[\varepsilon_3 - u]} \right). \tag{13}$$

The nine points P_i on C_0 are given by $P_i = P(\varepsilon_i)$ ($i = 1, \ldots, 9$). The variables $(f_1 : f_2 : f_3)$ then represent the homogeneous coordinates of the last point P_{10}.

The lattice P is interpreted as the Picard lattice of the rational surface X obtained by blowing-up $\mathbb{P}^2$ at the nine points P_i ($i = 1, \ldots, 9$). The metric $\langle \alpha, \beta \rangle$ is minus the intersection pairing. The null vector δ defined by $\delta = 3e_0 - \sum_{i=1}^{9} e_i$ represents the anti-canonical divisor of X. An element $L \in P$ defines a linear system $|L|$ on $\mathbb{P}^2$. For $L = de_0 - \sum_{i=1}^{9} m_i e_i \in P$, the linear system $|L|$, classically denoted by $C^d\big(P_1^{m_1} \cdots P_9^{m_9}\big)$ [11], represents a complete family of curves in $\mathbb{P}^2$ determined as the zero locus of the homogeneous polynomial with assigned degree d and zero multiplicity m_i at P_i ($i = 1, \ldots, 9$). The virtual genus $g(L)$ of a curve $C \in |L|$ and the virtual dimension $\dim|L|$ of the family $|L|$ is given by

$$2 - 2g(L) = \langle L, L - \delta \rangle \qquad 2\dim|L| = -\langle L, L + \delta \rangle \tag{14}$$

respectively. In the situation here, the above formulae give the real genus and dimension, respectively [12].

In our formulation, the Weyl group orbit $M = W\big(E_8^{(1)}\big) . e_1 \subset P$ plays an essential role. The formulae (14) imply that $g(\Lambda) = 0$ and $\dim|\Lambda| = 0$ for any $\Lambda \in M$. It then turns out that the τ function $\tau(\Lambda)$ is a homogeneous polynomial of degree d in the variables (f_1, f_2, f_3). Furthermore, the equation $\tau(\Lambda) = 0$ specifies the unique rational curve corresponding to the linear system associated with $\Lambda \in M$. Due to our normalization of the τ functions $\tau(\Lambda)$ in equations (6) and (9), we have

$$\tau(\Lambda)\Big|_{C_0} = [\Lambda - u] \prod_{i=1}^{9} [\varepsilon_i - u]^{m_i} \tau_0^d \prod_{i=1}^{9} \tau_i^{-m_i} \qquad (\tau_0 = \tau_1 \tau_2 \tau_3) \tag{15}$$

for $\Lambda = de_0 - \sum_{i=1}^{9} m_i e_i \in M$, under the parametrization (13) of C_0. We note that for a generic curve C_m of degree m, the $3m$ intersecting points $P(u_i) \in C_m \cap C_0$ ($i = 1, 2, \ldots, 3m$) satisfy the relation $\big[m\varepsilon_0 - \sum_{i=1}^{3m} u_i\big] = 0$.

As a typical case of proposition 2.2, our τ functions satisfy the following bilinear relation:

$$\begin{aligned}
&[\varepsilon_2 - \varepsilon_3][\varepsilon_0 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4]\tau(e_1)\tau(e_0 - e_1 - e_4) \\
&\quad + [\varepsilon_3 - \varepsilon_1][\varepsilon_0 - \varepsilon_3 - \varepsilon_1 - \varepsilon_4]\tau(e_2)\tau(e_0 - e_2 - e_4) \\
&\quad + [\varepsilon_1 - \varepsilon_2][\varepsilon_0 - \varepsilon_1 - \varepsilon_2 - \varepsilon_4]\tau(e_3)\tau(e_0 - e_3 - e_4) = 0.
\end{aligned} \tag{16}$$

This formula can also be proved geometrically as follows. For $L = e_0 - e_4$, we have $\dim|L| = 1$. Hence there must be a linear relation among the three terms. The coefficients can be determined by the normalization condition (15).

In the paper [15], a bilinear formulation of the elliptic Painlevé equation with $W\big(E_8^{(1)}\big)$ symmetry was proposed. Written in terms of the E_8 lattice, our bilinear relations coincide with those in [15]. This fact establishes the equivalence between the two formulations of the elliptic Painlevé equation.

3. Hypergeometric solutions

In order to determine the explicit form of the elliptic Painlevé equation, we consider the actions of translations in $W\big(E_8^{(1)}\big)$.

Lemma 3.1 [13]. *For a translation $T_\alpha \in W\big(E_8^{(1)}\big)$ along a vector in the root lattice $\alpha \in Q$, one has*[3]

$$T_\alpha(L) = L + k\alpha - \left(\frac{k}{2}\langle\alpha,\alpha\rangle + \langle L,\alpha\rangle\right)\delta \qquad k = \langle\delta, L\rangle. \tag{17}$$

The 240 roots of E_8 are represented as $\alpha_{ij} = e_i - e_j$ (72 vectors) or $\pm\alpha_{ijk} = \pm(e_0 - e_i - e_j - e_k)$ (2×84 vectors). The corresponding translations are given by

$$\begin{aligned} T_{\alpha_{ij}} &= s_{ia_1a_2}s_{ia_3a_4}s_{a_5a_6a_7}s_{ia_3a_4}s_{ia_1a_2}s_{ij} \qquad && \{i, j, a_1, \ldots, a_7\} = \{1, 2, \ldots, 9\} \\ T_{\alpha_{ijk}} &= s_{a_1a_2a_3}s_{a_4a_5a_6}s_{a_1a_2a_3}s_{ijk} && \{i, j, k, a_1, \ldots, a_6\} = \{1, 2, \ldots, 9\} \end{aligned} \tag{18}$$

where s_{ij} and s_{ijk} are the reflections with respect to the roots α_{ij} and α_{ijk} respectively. In what follows, we consider only the translations of type $T_{\alpha_{ij}}$. As an example, let us consider the translation

$$T_{\alpha_6} = s_{126}s_{346}s_{589}s_{346}s_{126}s_{67}. \tag{19}$$

To write down the explicit formula of the elliptic Painlevé equation with respect to the translation T_{α_6}, it is convenient to factorize the T_{α_6} as

$$T_{\alpha_6} = \mu^2 \qquad \mu = s_{126}s_{346}s_{568}s_{15}s_{28}s_{79}s_{67}. \tag{20}$$

The actions of μ on the parameters ε_i are given by

$$\begin{aligned} &\mu(\varepsilon_0) = \delta + \varepsilon_0 - 2\varepsilon_6 + \varepsilon_7 + \varepsilon_9 \qquad && \mu(\varepsilon_6) = \varepsilon_9 \\ &\mu(\varepsilon_7) = \delta - \varepsilon_6 + \varepsilon_7 + \varepsilon_9 && \mu(\varepsilon_9) = \varepsilon_7 \\ &\mu(\varepsilon_i) = \varepsilon_0 - \varepsilon_6 - \varepsilon_j \end{aligned} \tag{21}$$

where, in the last equation, $(i, j) = (1, 8), (2, 5), (3, 4), (4, 3), (5, 2), (8, 1)$. Here, by abuse of notation, we denote $\delta = 3\varepsilon_0 - \sum_{i=1}^{9}\varepsilon_i$. This δ appears only in the theta function $[x]$ and plays the role of the unit for the elliptic difference operation. We introduce intermediate variables g_i $(i = 1, 2, 3)$ by

$$g_i = \mu(f_i) = \frac{\tau(\mu s_0 \,.\, e_i)}{\tau(\mu \,.\, e_i)} = \frac{\tau(2e_0 - e_1 - e_2 - e_3 - e_6 - e_j)}{\tau(e_0 - e_6 - e_j)} \tag{22}$$

[3] A relation between the coefficient of e_0 and the algebraic entropy is pointed out in [14].

where $(i, j) = (1, 8), (2, 5), (3, 4)$. By a similar method in the proof of equation (16), one can show the following formulae for $i, j \geqslant 4$ $(i \neq j)$:

$$\begin{aligned}
&\tau(e_0 - e_i - e_j)\tau_i\tau_j = \frac{[1i][1j][1ij]}{[12][13][123]}\tau(e_0 - e_2 - e_3)\tau_2\tau_3 + (123 \text{ cyclic}) \\
&\tau(e_0 - e_1 - e_2 - e_3 - e_i - e_j)\tau_i\tau_j \\
&\qquad = -\frac{[23i][23j][1ij]}{[12][13][123]}\tau_1\tau(e_0 - e_1 - e_3)\tau(e_0 - e_1 - e_2) + (123 \text{ cyclic})
\end{aligned} \tag{23}$$

where $[ij] = [\varepsilon_i - \varepsilon_j]$ and $[ijk] = [\varepsilon_0 - \varepsilon_i - \varepsilon_j - \varepsilon_k]$. From these formulae we obtain

Proposition 3.2. *The action of T_{α_6} is given explicitly in the form*

$$g_i = \frac{Q_i(f)}{P_i(f)} \qquad \overline{f_i} = T_{\alpha_6}(f_i) = \frac{S_i(g)}{R_i(g)} \qquad (i = 1, 2, 3). \tag{24}$$

The polynomials P_i, Q_i, R_i and S_i are given by

$$\begin{aligned}
P_i &= \frac{[1j][16][1j6]}{[12][13][123]}f_1 - \frac{[2j][26][2j6]}{[12][23][123]}f_2 + \frac{[3j][36][3j6]}{[13][23][123]}f_3 \\
Q_i &= -\frac{[23j][236][1j6]}{[12][13][123]}f_2f_3 + \frac{[13j][136][2j6]}{[12][23][123]}f_1f_3 - \frac{[12j][126][3j6]}{[13][23][123]}f_1f_2 \\
R_i &= -\frac{[i8][689][i\overline{7}8]}{[48][58][123]}g_1 + \frac{[i5][569][i5\overline{7}]}{[45][58][123]}g_2 - \frac{[i4][469][i4\overline{7}]}{[45][48][123]}g_3 \\
S_i &= -\frac{[ab8][45\overline{7}][i\overline{7}8]}{[48][58][123]}g_2g_3 + \frac{[ab5][4\overline{7}8][i5\overline{7}]}{[45][58][123]}g_1g_3 - \frac{[ab4][5\overline{7}8][i4\overline{7}]}{[45][48][123]}g_1g_2
\end{aligned} \tag{25}$$

where $j = 8, 5, 4$ and $(a, b) = (2, 3), (1, 3), (1, 2)$ for $i = 1, 2, 3$, respectively, and $\varepsilon_{\overline{7}} = \varepsilon_7 + \delta$.

Proof. The first equation of (24) is a consequence of equations (22) and (23). The second equation is obtained by applying μ on the first one, namely, by replacing f_i with g_i, g_i with $\overline{f_i}$, and ε_i with $\mu(\varepsilon_i)$ of equation (21). □

Note that the system of difference equations (24) is written in terms of the homogeneous coordinates of $\mathbb{P}^2$. This system, apparently of third order, gives rise to a system of second order in the inhomogeneous coordinates.

To obtain the hypergeometric solution, let us put $[34] = 0$. Then we may consistently specialize the variables so that $\tau_3 = 0$, $f_3 = \infty$ and $g_3 = \infty$.[4] Then we have

$$\begin{aligned}
\overline{f_i} &= \frac{-[38][ab5][3\overline{7}8][i5\overline{7}]g_1 + [35][ab8][35\overline{7}][i\overline{7}8]g_2}{[58][i3][369][i3\overline{7}]} \\
g_i &= \frac{[13][13j][136][2j6]f_1 - [23][23j][236][1j6]f_2}{[12][3j][36][3j6]}
\end{aligned} \tag{26}$$

for $i = 1, 2$, with (a, b) and j as in proposition 3.2. Eliminating the variables g_i, we have

$$\overline{f_i} = \frac{a_i f_1 + b_i f_2}{c_i} \qquad (i = 1, 2) \tag{27}$$

[4] Another possibility is the case of $[124] = 0$ where the specialization $f_3 = g_3 = 0$ is available, which is the s_0 transform of $[34] = 0$, $f_3 = g_3 = \infty$. More generally, if $[ijk] = 0$ one can consistently specialize as $\tau(e_0 - e_i - e_j) = \tau(e_0 - e_j - e_k) = \tau(e_0 - e_k - e_i) = 0$. We will discuss the latter case in the next section with a different method.

where

$$
\begin{aligned}
a_i &= [13][136](-[ab5][3\overline{7}8][i5\overline{7}][356][138][268]+[ab8][35\overline{7}][i\overline{7}8][368][135][256])\\
b_i &= [23][236]([ab5][3\overline{7}8][i5\overline{7}][356][238][168]-[ab8][35\overline{7}][i\overline{7}8][368][235][156])\\
c_i &= [58][i3][369][i3\overline{7}][12][36][356][368].
\end{aligned}
\tag{28}
$$

From equation (27) and its $T_{\alpha_6}^{-1}$ transformation (obtained by replacing f_i, $\overline{f_i}$, $\varepsilon_{\overline{7}}$ and ε_6 with $\underline{f_i}$, f_i, ε_7 and $\varepsilon_{\underline{6}} = \varepsilon_6 + \delta$, respectively), we further eliminate f_2 and obtain a three-term relation for $\overline{f_1}$, f_1 and $\underline{f_1}$. We will show that this three-term relation can be identified with the difference equation for the elliptic hypergeometric function ${}_{10}E_9$. To this end, we apply the transformation $s_{289}s_{48}s_{39}$. Then we have the second-order equation for $F = s_{289}s_{48}s_{39}(f_1) = \tau_8/\tau_1$, under the specialization $[89] = 0$. Explicitly, it is given by

$$
\overline{F} = \frac{A_1}{A_2}F - \frac{B_1}{B_2}\underline{F}
\tag{29}
$$

where

$$
\begin{aligned}
A_1 = {} & [79][169][\underline{6}7](-[59][4\overline{7}9][246][346][149][569]+[49][5\overline{7}9][256][356][159][469])\\
& +[69][179][6\overline{7}]([49][5\underline{6}9][257][357][159][479]\\
& -[59][4\underline{6}9][247][347][149][579])\\
A_2 = {} & [199][45][\underline{6}7][1\overline{7}9][79]\prod_{k=2}^{5}[k69]\\
B_1 = {} & [6\overline{7}][1\underline{6}9][169][69]\prod_{k=1}^{5}[k79] \qquad B_2 = [\underline{6}7][1\overline{7}9][179][79]\prod_{k=1}^{5}[k69].
\end{aligned}
\tag{30}
$$

Further, we apply a gauge transformation $F = g\Phi$ by using a solution g of the difference equation

$$
\overline{g} = \frac{[169][\overline{7}99]}{[1\overline{7}9][699]}g.
\tag{31}
$$

Then we finally obtain the following:

Theorem 3.3. *The linear difference equation for Φ is determined as*

$$
\begin{aligned}
&\frac{[79][\overline{7}99][799]\prod_{k=1}^{5}[k69]}{[76][\overline{7}6]}(\overline{\Phi}-\Phi)\\
&\quad = \frac{[69][\underline{6}99][699]\prod_{k=1}^{5}[k79]}{[76][7\underline{6}]}(\Phi-\underline{\Phi}) - [679]\prod_{k=1}^{5}[k9]\Phi.
\end{aligned}
\tag{32}
$$

We introduce the parameters u_i $(i = 0, 1, \ldots, 7)$ by setting

$$
u_0 = \varepsilon_0 - 3\varepsilon_9 - \delta \qquad u_i = \varepsilon_i - \varepsilon_9 \qquad (i = 1, \ldots, 7).
\tag{33}
$$

Note that the parameters u_i satisfy the balancing restriction

$$
2\delta + 3u_0 - \sum_{i=1}^{7} u_i = 0.
\tag{34}
$$

Then the difference equation (32) for Φ can be rewritten as the *elliptic hypergeometric equation*

$$\begin{aligned}
&\frac{[u_7][u_0-u_7][\delta+u_0-u_7]\prod_{i=1}^{5}[\delta+u_0-u_i-u_6]}{[u_7-u_6][u_7-u_6+\delta]}[\Phi(6^-7^+)-\Phi]\\
&\qquad=\frac{[u_6][u_0-u_6][\delta+u_0-u_6]\prod_{i=1}^{5}[\delta+u_0-u_i-u_7]}{[u_7-u_6][u_7-u_6-\delta]}[\Phi-\Phi(7^-6^+)]\\
&\qquad\quad+\prod_{i=1}^{5}[u_i][u_7+u_6-u_0-\delta]\Phi
\end{aligned}\tag{35}$$

where $\Phi(i^-j^+) = \Phi|_{u_i\mapsto u_i-\delta, u_j\mapsto u_j+\delta}$. It is known [6] that if one of the parameters u_i $(i = 1,\dots,7)$ is $0, -\delta, -2\delta, \dots,$ then the terminating very-well-poised balanced elliptic hypergeometric series

$$\begin{aligned}
&\Phi = {}_{10}E_9(u_0,u_1,\dots,u_7) = \sum_{n=0}^{\infty}\frac{[u_0+2n\delta]}{[u_0]}\prod_{r=0}^{5}\frac{[u_r]_n}{[u_0-u_r+\delta]_n}\\
&[z]_n = [z][z+\delta]\cdots[z+(n-1)\delta]
\end{aligned}\tag{36}$$

solves the linear equation (35). Hence, the elliptic hypergeometric function ${}_{10}E_9$ gives a particular solution of the elliptic Painlevé equation. As a byproduct, we find that ${}_{10}E_9$ has the affine Weyl group symmetry of type $E_7^{(1)}$.

4. Geometric formulation

In this section, we give a geometric interpretation of the results in the previous section.

When the nine points $P_1,\dots,P_9$ are in special positions such that they are the base of an elliptic fibration $X \to \mathbb{P}^1$, the translation subgroup $\mathbb{Z}^8 \subset W\big(E_8^{(1)}\big)$ is realized as the Mordell–Weil group of the elliptic fibration. This fact follows from [16], theorems 5, 6. In the case of the Painlevé equation, the configuration of the points $P_1,\dots,P_9$ is not special in the sense above. However, for the translations of the type $T_{\alpha_{ij}}$, one can also find the special configuration at the intermediate step of the translation, and the above correspondence between the translations and Mordell–Weil group is applicable also for the Painlevé equation.

Let us again consider the translation T_{α_6} as an example. Under the translation T_{α_6}, the points P_i $(i \neq 6, 7, 10)$ are invariant and the new points $\overline{P}_6$ and $\overline{P}_7$ are determined so that

$$P_6+P_7=\overline{P}_6+\overline{P}_7 \qquad P_1+\cdots+P_6+\overline{P}_7+P_8+P_9=0\tag{37}$$

with respect to the addition on the cubic C_0. This means that $\overline{P}_7$ is the additional intersection point of the elliptic pencil defined by the eight points P_i $(i \neq 7, 10)$. Using this pencil of cubics, the transformation $T_{\alpha_6}(P_{10})$ is geometrically described as follows. Consider a cubic curve C passing through the nine points P_i $(i \neq 7)$. The new point $\overline{P}_{10}$ is determined by

$$\overline{P}_{10}+\overline{P}_7=P_{10}+P_6\tag{38}$$

with respect to the addition on the curve C. In this sense, the elliptic Painlevé equation is a non-autonomous deformation of the addition formula of the elliptic function, where the addition is defined on the moving curve C.

Let us apply this formulation to construct the hypergeometric solutions. To do this, consider the case when the three points P_4, P_5, P_9 are on a line ℓ. In such a case, if $P_{10} \in \ell$ then $\overline{P}_{10} \in \ell$, since the curve C is decomposed into the line ℓ (passing through $P_4, P_5, P_9, \overline{P}_{10}$) and the conic C_2 (passing through $P_1, P_2, P_3, P_6, \overline{P}_7, P_8$). Let us put

$$x=P_6 \qquad \overline{y}=\overline{P}_7 \qquad f=P_{10} \qquad \overline{f}=\overline{P}_{10}\tag{39}$$

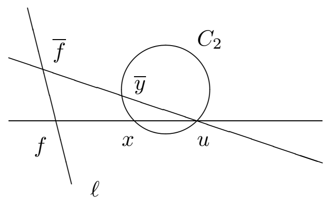


Figure 1. Configuration of the points.

then these points are in the configuration shown in figure 1.

By elementary geometry we have

Lemma 4.1. *f satisfies the linear difference equation*

$$\begin{aligned}(a,x)\overline{f} &= (a,\overline{y})Df - (a,Df)\overline{y}\\ (a,\overline{y})f &= (a,x)D^{-1}\overline{f} - (a,D^{-1}\overline{f})x\end{aligned} \tag{40}$$

where $D = \mathrm{diag}\left(\frac{x_2x_3}{\overline{y}_2\overline{y}_3}, \frac{x_3x_1}{\overline{y}_3\overline{y}_1}, \frac{x_1x_2}{\overline{y}_1\overline{y}_2}\right)$, $(a,x) = a_1x_1 + a_2x_2 + a_3x_3 = 0$ *is the equation of the line* ℓ, *and* $x_i, \overline{y}_i$ $(i = 1,2,3)$ *are homogeneous coordinates for* x *and* $\overline{y}$, *respectively.*

The normalization of the variables f is chosen so that $f_i = \tau(s_0 . e_i)/\tau(e_i)$. From equation (40), it is easy to deduce the following:

Proposition 4.2. *f_1 satisfies the linear difference equation*

$$\frac{y_1\overline{y}_3(a,x)[\overline{y}_2\overline{f}_1 - x_2f_1]}{(x_2\overline{y}_3 - x_3\overline{y}_2)} - \frac{x_1\underline{x}_3(a,y)[\underline{x}_2\underline{f}_1 - y_2f_1]}{(\underline{x}_2y_3 - \underline{x}_3y_2)} = a_2(x_1y_2 - x_2y_1)f_1. \tag{41}$$

Using the parametrization of the points P_i $(1 \leqslant i \leqslant 9)$ by the theta function $P_i = \left(\frac{[23i]}{[1i]} : \frac{[13i]}{[2i]} : \frac{[12i]}{[3i]}\right)$, we finally obtain the following:

Theorem 4.3. *When the points P_4, P_5, P_9 and P_{10} are on a line, the elliptic Painlevé equation for the translation T_{α_6} is reduced to the linear difference equation*

$$c_1\left(\frac{[26][13\overline{7}]}{[2\overline{7}][136]}\overline{f}_1 - f_1\right) - c_2\left(\frac{[27][13\underline{6}]}{[2\underline{6}][137]}\underline{f}_1 - f_1\right) = c_3f_1 \tag{42}$$

where

$$\begin{aligned}c_1 &= [27][2\overline{7}][46][56][12\overline{7}][136][237][456]/[6\overline{7}]\\ c_2 &= [26][2\underline{6}][47][57][12\underline{6}][137][236][457]/[7\underline{6}]\\ c_3 &= [24][25][76][123][1\overline{7}6][245][376]\\ \varepsilon_{\overline{7}} &= \varepsilon_7 + \delta \qquad \varepsilon_{\underline{6}} = \varepsilon_6 + \delta.\end{aligned} \tag{43}$$

By suitable gauge transformation and parameter change, equation (42) also coincides with equation (35).

The above geometric formulation can also be applied for the degenerate cases. In particular, when the cubic C_0 factors into a conic and a line (resp. three lines), then the symmetry reduces to $W\left(E_7^{(1)}\right)$ (resp. $W(E_6^{(1)})$) and the corresponding hypergeometric solution is reduced to the Askey–Wilson (resp. big q-Jacobi) functions.

We believe that our results will open a new avenue to the systematic study of the special functions (Askey scheme) from the point of view of the affine Weyl group symmetry.

Acknowledgments

We would like to thank M-H Saito, V P Spiridonov and A S Zhedanov for valuable discussions.

References

[1] Sakai H 2001 *Commun. Math. Phys.* **220** 165–221
[2] Iwasaki K, Kimura H, Shimomura S and Yoshida M 1991 *From Gauss to Painlevé* (Braunschweig: Vieweg)
[3] Ramani A, Grammaticos B, Tamizhmani T and Tamizhmani K M 2001 *Comput. Math.* **42** 603–14
[4] Murata M, Sakai H and Yoneda J 2003 *J. Math. Phys.* **44** 1396–414
[5] Frenkel I B and Turaev V G 1997 *Arnold–Gelfand Mathematical Seminars* (Boston, MA: Birkhauser) pp 171–204
[6] Spiridonov V P and Zhedanov A S 2001 *Special Functions 2000* (Dordrecht: Kluwer) pp 365–88
[7] Okamoto K 1979 *Japan. J. Math.* **5** 1–79
[8] Sioda T and Takano K 1997 *Funkcial. Ekvac.* **40** 271–91
[9] Dolgachev I and Ortland D 1988 *Astérisque* **165**
[10] Mizoguchi S and Yamada Y 2002 *Phys. Lett.* B **537** 130–40
[11] Semple J G and Roth L 1949 *Introduction to Algebraic Geometry* (Oxford: Clarendon)
[12] Nagata M 1960 *Mem. Coll. Sci. Univ. Kyoto* **38** 271–93
[13] Kac V G 1990 *Infinite Dimensional Lie Algebras* 3rd edn (Cambridge: Cambridge University Press)
[14] Takenawa T 2001 *J. Phys. A: Math. Gen.* **34** 10533–45
[15] Ohta Y, Ramani A and Gramaticos B 2001 *J. Phys. A: Math. Gen.* **34** 10523–32
[16] Manin Ju I 1964 *Izv. Akad. Nauk SSSR Ser. Mat.* **28** 1363–90
Manin Ju I 1966 *AMS Transl.* **59** 82–110